MODELLING OF ELASTO-HYDRODYNAMIC LUBRICATION PROBLEMS IN GEARS

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Abstract

For highly loaded contacts, lubricant pressure can cause elastic deformation of the surface that is in the same order as the film thickness of lubricant. When this occurs, the influence of deformation on lubrication performance becomes an important parameter. Contacts operating under this condition are in the regime of elasto-hydrodynamic lubrication (EHL).In addition, numerical calculations of EHL problems are high-priced belonging to the large quantity of numerical computations needed as a result of their contact domains being much bigger than those of involute gears and bearings [1,2].

To amplify the effortlessness of simulations, Homotopy Perturbation Method (HPM)[3-6,13-17] *is used. HPM is an effective and relatively uncomplicated methodology for solving nonlinear thermal problem for lubrication of gears. In this paper, we contrast HPM in related to higher-order approximate solutions for the EHL. It is concluded that the results of HPM provide accurate solution, as an easy methodology for computing the nonlinear differential equations for EHL.*

KEYWORDS : *Elasto-hydrodynamic, Gear lubrication , Homotopy Perturbation Method*

INTRODUTION

One of the most perplexed applications of tribology is the tooth contact of gears. For design of gears, there is no suitable replacement for virtual occurrence. In association with the many types of gears, the most commonly used are spur gears. Because loads distribution on the gears are channelized through lubricated contacts of Hertzian as elasto-hydrodynamic (EHL) contacts, the action of lubrication in spur gears has a large extent conjointly [7,8].

Because of the elastic deformation, the results in high contact pressures due to the contacting lubricated domains in gears will be generated by the high pressures in contact zones [9].

Uncomplicated and simple and direct numerical solutions for solving of line contact problems in gears cannot establish high convergence at the position of loads distribution [10].

There are some numerical methodologies applied comprehensively for providing the results of EHL problems such as : inverse methods, forward-iterative methods, multi-grid methods and Newton-Raphson methods. In the group of these methods, the Newton-Raphson methodology conventionally obtains a methodical approach and a better coincidence in nonlinear system investigation [11].

Because application of the Reynolds equation as the main equation in EHL problem, and also lack of analytical solution, the Reynolds equation frequently solved numerically. Some of iterative methodologies are comparatively simple to perform than straightforward methods. Due to complexity of Reynolds equation as a second-order nonlinear partial differential equation, it is difficult to be solved by iterative methods [12].

Many analytical methods have been used to solve nonlinear problem in elasto-hydrodynamic lubrication. For example, perturbation methods [2,3] are mentioned to be the most common tools in nonlinear analysis of engineering problems. These methods are continuously being amplified and applied to ever more complicated problems. However, traditional perturbation methods have many shortcomings and are not valid for strongly nonlinear equations. To overcome the shortcomings, many new techniques have been proposed in the literature, for example: Homotopy Perturbation Method [4-6,13-17], Lindstedt-Poincaré [18], Harmonic Balance Method [19], Variational Iteration Method [20-21], and Variational Approach Method [3,22].

The main focus of this paper is on a comparative study of application of the Homotopy Perturbation Method (HPM) with other available techniques in solution of nonlinear thermal problem of a elasto-hydrodynamic lubrication in gears.

We use Homotopy perturbation method (HPM) to observe the approximate analytical solution of nonlinear differential equation of elasto-hydrodynamic lubrication in gears. In the numerical methodologies, stableness and coincidence should be considered so as to keep off abnormality or incompatible results. With the formulation of the elasto-hydrodynamic lubrication problem it can be supported that the system of equations can be cut down to some parameter problem by presenting appropriate transformation variables.

FUNDAMENTAL OF THE HOMOTOPY BERTURBATION METHOD (HPM)

HPM is combined by the classical perturbation technique and homotopy technique [3-4,13-17] which removes the restriction of the classical perturbation methods. For explanation of the basic idea for the presented method as a method to solve the nonlinear differential equations, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \; r \in \Omega \tag{1}$$

where A is operator, f is a known function and u is a sought function. Assume that operator A can be written as:

$$A(u) = L(u) + N(u), \quad r \in \Omega, \tag{2}$$

where L is the linear operator, N is the nonlinear operator. Hence, Eq. (1) can be rewritten as follows:

$$L(u)+N(u)=f(r), \quad r\in\Omega, \tag{3}$$

In case the nonlinear Eq. (1) has no small parameter, We define an operator *H* as:

$$H(v,\varepsilon)=(1-\varepsilon)\left[L(v)-L(u_0)\right]+\varepsilon\left[A(v)-f(r)\right]=0 \tag{4}$$

Where $v(r,\varepsilon)$: $\Omega\times[0,1]\rightarrow R$ and in Eq. (4), $\varepsilon\in[0,1]$ is called homotopy parameter, and u_0 is the first approximation that satisfied the boundary condition. According to the HPM, the assumption of approximation solution of Eq. (4) can be expressed as a power series of *p*:

$$v=v_0+\varepsilon v_1+\varepsilon^2 v_2+\varepsilon^3 v_3+\ldots \tag{5}$$

Setting $\varepsilon=1$ result in the approximate solution for Eq. (4) and eventually the best approximation for solution is:

$$u=\lim_{p\rightarrow 1} v=v_0+v_1+v_2+\ldots \tag{6}$$

Convergence of the Eq. (6) is happened for most of the cases, and also the rate of convergence is related to the selection of operator *A*.

The LUNRICATION MODEL

There is a great deal alteration in the pressure in EHL lubricant problem over very small length, therefore using a special accurate model for lubrication can be most important. Due to this significance many researcher are currently working on the model to analyze the behavior of oil in during the lubricant process.

The Reynolds equation

For the distribution of pressure because of an applied load for a special geometry the Reynolds equation defined that was derived from the equations of Navier-Stokes. The Reynolds equation in Cartesian coordinates and steady-state situation is:

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{12\eta}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial}\left(\frac{\rho h^3}{12\eta}\frac{\partial p}{\partial y}\right)=\frac{\partial}{\partial x}(\rho\overline{u}h)+\frac{\partial}{\partial y}(\rho\overline{w}h) \tag{7}$$

Where *p* is the pressure, h is film thickness, η is the viscosity of the lubricant, ρ is the density of the lubricant, *x* is the coordinate at rolling direction [10] , $\overline{u}=(u_a+u_b)/2$ is velocities of lubricant because of speeds of surface *a* and *b* in *x* direction and also $\overline{w}=(w_a+w_b)/2$ is velocities of lubricant because of speeds of surface *a* and *b* in *y* direction.

Reynolds investigation in second order

The classical investigation to the second-order solvation of the steady-state isothermal line contact problem requires some equations. The line contact hydrodynamic in only steady-state results is:

$$\frac{\partial}{\partial X}\left(\frac{\rho h^3}{12\eta}\frac{\partial P}{\partial X}\right)=\overline{u}\frac{\partial}{\partial X}(\rho h) \tag{8}$$

The relationship between density-pressure

Pressure and temperature influence on viscosity are significant. With any changes in pressure, the alteration of density is small with regard to the change in viscosity quantity. It can be written for dimensionless density as:

$$\rho=\rho_0\left(1+\frac{0.6p}{1+1.7p}\right) \tag{9}$$

where ρ_0 is the density at zero pressure.

The relationship between viscosity-pressure

The viscosity of lubricant is very important in EHL contacts. Barus' equation introduces the relation of viscosity on pressure that is:

$$\eta = \eta_0 \exp(\alpha p) \tag{10}$$

where η_0 is the absolute viscosity at a constant temperature and at p_0, and the parameter of α is the coefficient of pressure-viscosity in lubricant and it is temperature dependent.

The formulation of Elasticity equation

Because of the pressure which is grew in the fluid of lubricating, the elastic equation defined as normal displacements to the domains. The total film thickness is represented by:

$$h = h_0 + \frac{x^2}{2R} + \sum_{j=1}^{n} f_{ij} * p_j \tag{11}$$

where h_0 is the initial film thickness, $x^2/2R$ is the separation due to the geometry, where and R is the radius of relative curvature equivalent radius defined by $1/R = 1/R_1 + 1/R_2$ here R_1, R_2 represent the radius of curvature of the intersecting areas and the last part of the equation, $\sum_{j=1}^{n} f_{ij} * p_j$ as the total elastic deformation of the surface is $-\frac{2}{\pi E'}\int_{x_{in}}^{x_{out}} p(x')\ln(x-x')^2 dx'$.

Non-dimensionalisation

Non-dimensionalisation is recommended for solving the numerically calculation. It permits the coincidental modelling of all kind of numerical solving and lends to wide generality to the results [1,23-24]. The Reynolds equation for thermal and steady state conditions in non-dimension model can be written as [11]:

$$\frac{\partial}{\partial x}\left(\frac{\bar{\rho} h^3}{\bar{\eta}}\frac{\partial p}{\partial x}\right) - 12u_m \frac{\partial}{\partial x}\left(\bar{\rho} h\right) = 0; \qquad , u_m = \frac{u_a + u_b}{2} \tag{12}$$

with these dimensionless parameters:

$$P = \frac{p}{p_H}; X = \frac{x}{b}; H = \frac{hR}{b^2}; \rho = \frac{\bar{\rho}}{\rho_0}; \eta = \frac{\bar{\eta}}{\eta_0}; H_0 = \frac{h_0 R}{b^2}; \tag{13}$$

That b is half Hertzian contact length, p_H is maximum Hertzian pressure, and finally dimensionless Reynolds equation can be rewritten as:

$$\frac{\partial}{\partial X}\left(\frac{\rho H^3}{\eta}\frac{\partial P}{\partial X}\right) - \lambda \frac{\partial}{\partial X}\left(\rho H\right) = 0; \quad , \lambda = \frac{12\eta_0 u_m R_x^2}{b^3 P_H} \tag{14}$$

With the following boundary conditions:

$$x = x_{in}, p = 0; \quad and \quad x = x_{out}, p = \partial p / \partial x = 0 \tag{15}$$

And also h_d as elastic deformation at any point x on the area is determined by:

$$h_d(x) = -\frac{2}{\pi E'}\int_{x_{in}}^{x_{out}} p(x')\ln(x-x')^2 dx' \tag{16}$$

E' is the equivalent Young's modulus given by:

$$\frac{2}{E'} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \tag{17}$$

Where E_1, E_2 the elastic moduli and ν_1, ν_2 are Poisson ratios of the two surface materials. that dimensionless form is:

$$H_d(x) = -\frac{1}{\pi}\int_{X_{in}}^{X_{out}} P(X')\ln(X-X')^2 dX' \tag{18}$$

The film thickness of EHL for line contact is:

$$h(x) = h_0 + h_g(x) + h_d(x) \tag{19}$$

Thus, for any point X_i the shape of film is given by:

$$H(X_i) = H_0 + \frac{X_i^2}{2} - \frac{1}{\pi}\int_{X_{in}}^{X_{out}} P(X')\ln(X-X')^2 dX' \tag{20}$$

H_0 contains the constant term of $H_d(x)$ as initial film thickness. Film thickness equation is represented as:

$$H(X_i) = H_0 + \frac{X_i^2}{2} - \frac{1}{\pi}\sum_{k=1}^{M} D_{i,k}P_k \tag{21}$$

Solution to EHL by Homotopy perturbation method (HPM)

For the values of parameters in the presented equation, we use references [2,7,11].

Let us construct the homotopy map for Eq. (14), as follows:

$$H(P,\varepsilon) = (1-\varepsilon)\left(0.001284737157\left(\frac{d^2}{dx^2}\mathrm{P(x)}\right)-0.0086\mathrm{x}+0.001191256\left(\frac{d}{dx}\mathrm{P(x)}\right)+0.005402454386\left(\frac{d}{dx}\mathrm{P(x)}\right)x\right)$$
$$+\varepsilon\left(5562.7\left(0.36\left(\frac{d}{dx}\mathrm{P(x)}\right)-0.7344\mathrm{P(x)}\left(\frac{d}{dx}\mathrm{P(x)}\right)+\ldots\right.\right. \tag{22}$$

Substituting in $P(x) = P_0(x) + \varepsilon P_1(x) + \varepsilon^2 P_2(x)$ as pressure to Eq. (22) and rearranging Eq.(22) based on powers of å-terms, we have:

$$\varepsilon^0 : 0.005402454386\left(\frac{d}{dx}\mathrm{P_0(x)}\right)\mathrm{x} + 0.001191256\left(\frac{d}{dx}\mathrm{P_0(x)}\right) - 0.0086\mathrm{x} + 0.001284737157\left(\frac{d^2}{dx^2}\mathrm{P_0(x)}\right) \tag{23}$$

$$\varepsilon^1 : -0.1717461481(\mathrm{P_0(x)})^8\left(\frac{d^2}{dx^2}\mathrm{P_0(x)}\right)x^2 + 0.02134253067(\mathrm{P_0(x)})^6\left(\frac{d^2}{dx^2}\mathrm{P_0(x)}\right)x^6 + 0.005402454386\left(\frac{d}{dx}\mathrm{P_1(x)}\right)\mathrm{x}+\ldots \tag{24}$$

$$\varepsilon^2 : -0.04188881207\left(\frac{d}{dx}\mathrm{P_1(x)}\right)\left(\frac{d}{dx}\mathrm{P_0(x)}\right) + 0.3391522094\left(\frac{d}{dx}\mathrm{P_0(x)}\right)^2\mathrm{P_1(x)} + 0.00695383776\left(\frac{d}{dx}\mathrm{P_1(x)}\right)\mathrm{P_0(x)}+\ldots \tag{25}$$

With the boundary conditions and in the same manner, the rest of components were obtained using the maple package.

According to the HPM, we can conclude that:

$$\mathrm{P}_{\mathrm{Pr}essure} = \lim_{\varepsilon\to l} P(x) = P_0(x) + P_1(x) + P_2(x) + \ldots \tag{26}$$

Therefore, substituting the values of $P_0(x)$, $P_1(x)$ and $P_2(x)$ from Eqs. (23,24,25) in to Eq. (26) yields:

$$P(x) = 0.0115x + 0.01725 + 0.1439705088(x+1.5)^2 10^{24} - 0.7599551677(x+1.5)^3 10^{24}$$
$$+ 0.39406754470(x+1.5)^4 10^{25} - 0.1447641698(x+1.5)^5 10^{26}$$
$$+ 0.4310979279(x+1.5)^6 10^{26} + O\left((x+1.5)^7\right) \tag{27}$$

The results are shown in figures 1 and 2, where $P = p/p_H$ is as dimensionless pressure and $X = x/b$ is as dimensionless location.

For the accuracy estimation of the approximate analytical approach, a comparison between HPM and a few other methods in the indicated references [7,9].Comparing the results, HPM presents a good result and its power and capability is one of the advantages of this method.

In this work, MAPLE package was used in the mathematical calculations.

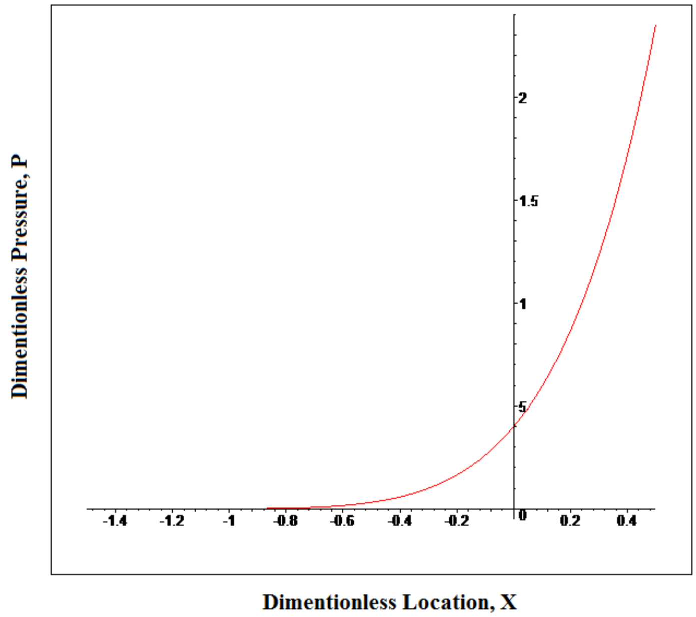

Figure 1 : *Pressure distributions obtained using HPM methods across the entire contact.*

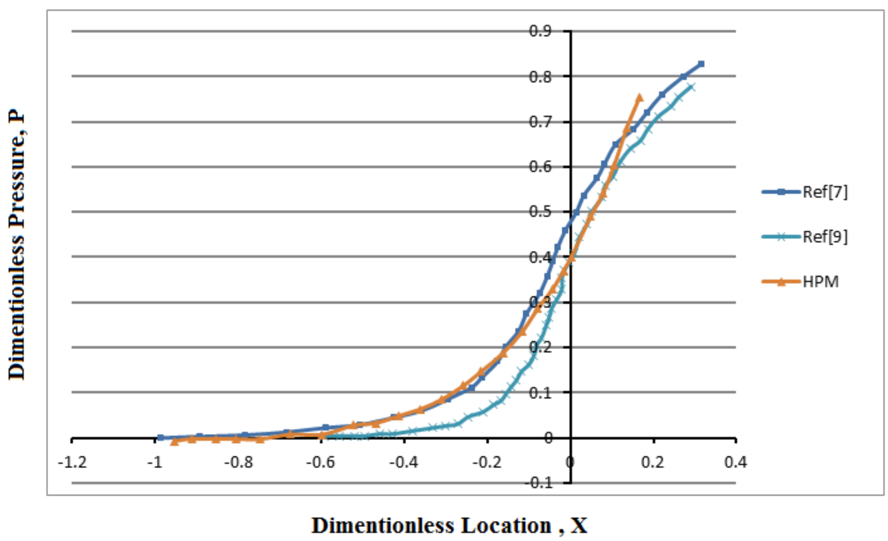

Figure 2 :*Comparison with Data from Ref.[7,9].*

CONCLUSION

In this study, one of the fast computational methodology as Homotopy Perturbation Method (HPM) was successfully utilized to solve the EHL problems. In this work, HPM is applied for the analysis to solve the Reynolds equation. The achieved results indicate that the present method can be easily extended to the analysis of two-dimensional thermal and isothermal EHL problems such as finite line contact and point contact problems.
The results are compared with a few other numerical methods [7,9] and show good correspondence. Applying HPM to Reynolds equation has some advantages comparing to the other methods: it overcomes the difficulties arising in the calculation Newton-Raphson

method. Also HPM does not require small parameters such as classical perturbation techniques in the equation, so that the limitations of the traditional perturbation methods can be eliminated, and thereby the calculations are simple and straightforward.

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